TIME–FREQUENCY ANALYSIS OF ARTERIAL PRESSURE OSCILLATIONS IN ANESTHETIZED DOGS: EFFECTS OF STANDARIZED HEMORRHAGES

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Received 29 Oct 1999; first review completed 19 Nov 1999; accepted in final form 6 Jun 2000

ABSTRACT—The purpose of this preliminary study was to investigate the advantages of the time–frequency analysis through the Continuous Wavelet Transform (CWT) compared to classical Fourier analysis using the Fast Fourier Transform (FFT) in arterial pressure signals from anesthetized mongrel dogs before and during standardized hemorrhages. Systemic arterial pressure pulsations were recorded using catheter-tip manometers. CWT and FFT were applied to arterial pressure pulsations to obtain module coefficients of this transformation and its associated contours during the evolution of progressive hemorrhages, in amounts of 15, 34, and 66% of the estimated total blood volume. This mathematical analysis enabled us to identify the evolution of the frequency components of aortic valve functions, heart dynamics, respiratory influences, and vasomotor activities. Furthermore, we isolated the modulating signal of amplitude modulation phenomenon present in the arterial pressure records, as described in previous papers, being the heart rate carrier frequency. The CWT is a very sensitive and reliable procedure to analyze (time–frequency) the oscillatory phenomena in two dimensions, and to provide more information than the FFT. This new analytical procedure may provide new insights in the study of shock pathophysiology.

KEYWORDS—Pressure pulsations, standardized hemorrhages, continuous wavelet transform, modulating signal, Fourier spectrum

INTRODUCTION

Because blood is the main physiological transport system of respiratory gases, metabolites, hormones, vitamins, and the agents of humoral and cellular immunity, any reduction of blood volume will have a great impact on the whole organism, even when bleeding is performed under general anesthesia applying non-traumatic procedures (1).

Despite a large number of studies performed on shock and hemorrhage pathophysiology, some aspects remain unclear, especially those related to the cardiovascular response to blood volume decrease and its relationship with the reversibility and non-reversibility of this phenomenon (2).

One of the classic methods for the study of these cardiovascular signals uses the Fast Fourier Transform (FFT), which is a one-dimensional analysis (only frequency) (3–5). We have developed and adapted algorithms based on wavelet theory, allowing two- or three-dimensional graphic representations (time–frequency plane or time–frequency–amplitude), that correspond to the Continuous Wavelet Transform (CWT). This methodology has been applied to cardiovascular signals under different hemodynamic conditions. The CWT sensitivity and precision was studied in previous papers (6–8), assuring us that the same method could be useful in the study of the cardiovascular response to hemorrhage.

This is an exploratory study that allows us to investigate the advantages of time–frequency analysis given by CWT when compared to the classic Fourier analysis using data from hemodynamic responses of the systemic circulation to a progressive bleeding procedure. In addition, we are interested in studying, under these conditions, an amplitude modulation phenomenon (AMP) detected by Günther et al. (6), where heart rate is the carrier frequency.

The present study does not correspond to a specific research on "shock," but to progressive hemorrhages that were realized in deeply anesthetized mongrel dogs. We intended to examine the pathway to an hypovolemic shock syndrome, with the aim to discover some premorbid indices using the CWT of arterial pulsations, more precisely at the transition from "reversible" to "irreversible" shock.

At the present time, this novel analysis can be made a posteriori due to the highly time-consuming computer procedures involved. Real time of CWT in medicine will come later when the CWT analysis could be performed at higher speeds, at which time they will then be useful for instantaneous decision regarding diagnosis, prognosis, and rational treatment of clinical shock.

MATERIALS AND METHODS

Experiments were conducted in anesthetized mongrel dogs of either sex, weighing 10–15 Kg. Sodium pentobarbital (30 mg/kg) was administered intravenously (i.v.) as a bolus. The trachea of all animals was intubated to assure a permeable upper airway during spontaneous breathing. A catheter with a micro-tip pressure transducer (model SPG-350, Millar Instruments, Houston, TX USA) was introduced into the left femoral artery until the micro-tip...
reached the thoracic aorta at the heart level (the distance between femoral and carotid arteries was previously determined). One hour of halothane (5600 U) was administered i.v. after the arterial cannulation. Another polyethylene catheter was introduced into the abdominal aorta through the femoral artery to allow bleeding. Experiments were carried out in the Experimental Surgical Laboratory of the Faculty of Medicine, University of Concepción, Chile, according to the rules of the National Institute of Health Guidelines for Care and Use of Laboratory Animals, USA. All these experiments were approved by the Ethic Committee of the Faculty of Medicine, by the Research Board of the University of Concepción, Chile, and by the National Commission of Scientific Research of Chile (CONICYT).

The estimated total blood volume of mammals is equivalent to 6% of the body weight, and consequently, the magnitude of each bleeding episode was expressed as a percentage of the estimated total blood volume. The allometric equation (r = 0.97) for the total blood volume is \( V = 0.0076 W^{0.97} \) (1), with \( r = 0.97 \). This equation is valid for all mammals, irrespective of their sizes, from 0.1 to 1000 kg (5).

**Bleeding and blood pressure**

Bleeding and blood pressures were recorded according to the following protocol, based on a previous study (10).

**First stage**—Control conditions were recorded before starting with the procedure, in the unanesthetized and heparinized dog (Fig. 1). The spontaneous bleeding was then initiated through cannulated femoral artery reaching 15% of the estimated blood volume (EBV). The arterial blood pressure was continuously monitored and recorded, and arterial pressure records analyzed using CWT and FFT in two steps. From 0 to 6% of EBV (Fig. 2), and from 6 to 15% of EBV (Fig. 3).

**Second stage**—Thirty minutes after the first hemorrhage was concluded, a new spontaneous bleeding was initiated and maintained until reaching 45% of EBV, the double of the first bleeding (approximated). The blood pressure recorded after the bleeding is shown in Figure 4.

**Third stage**—Thirty minutes after the second bleeding, a hemorrhage was forced (using syringe) duplicating the previous bleeding, obtaining a final loss of approximately 66% of EBV (Fig. 5).

**Data processing**

Analogue signals of arterial pressure oscillations were sampled using an Opus 22 (LC4) single-channel ana/dig converter at a rate of 42.7 samples per second, the data were fed into a personal computer and displayed by the software LVIEW (National Instruments, Austin, TX USA).

Power spectrum was calculated considering a number of harmonics within a scale frequency from 0 to 10 Hz. The frequencies outside this range were considered irrelevant for this study. The corresponding computational procedures were developed by biospecs (8). The CWT yielded a two-dimensional image called wavelet coefficient module, which represents a surface where the axes are time and frequency. A third dimension corresponds to the coefficient amplitude (module), and it is represented by a color palette where blue–white is the minimal value and purple–red is the maximal value. The values of the wavelet coefficients are relative to the signal under study and have no physical meaning units and provide information regarding the amount of energy contained in the signal (8). In the figures, we present the contours of the module corresponding to the level curves of the surface of the coefficients. The vertical axis, where frequencies are measured, is a logarithmic scale. The horizontal axis is a date of the frequency scale, given by Fourier spectral octave = logarithm on base 2 of frequency, i.e., 2^1 = 2 Hz (11).

In the wavelet analysis (11–13) the frequency axis should be measured in octaves (octave is a musical scale), and not in Hz. This measure is sub-divided into voices. We have been working with a frequency band of 10 octaves, with 8 voices each. With such values we obtained an image, from where the name "continuum" comes for the CWT.

The numerical output of applying CWT to a signal is a complex matrix of wavelet coefficients, which dimension is given from the sampling points (the horizontal axis representing the time) and the total number of voices (vertical axis). Every complex number admits a module and a phase and provided that it has no linear graphical interpretation, in this work we have only considered the module.

The modulating signal (MS) associated to the previously described AM phenomenon was obtained by a mathematical procedure, which considers only the maximum values of the wavelet coefficient modules, in the interval corresponding to the heart rate frequency (HR). This MS represents the relative power of the heart (work per unit time) at every moment, without a quantitative physical meaning.

**RESULTS**

**Blood pressure analysis under control conditions**

Figure 1 shows a graph of the original arterial pressure record, the modulated signal (MS), the Fourier spectrum, and the contour of module given by CWT under control conditions. In Figure 1A, the record of the systemic arterial pressure oscillations (AOP, black line) and the MS (red line) are shown. It can be seen that breathing movements have a marked influence on the AOP; however, no amplitude-modulated phenomena (AMP) are visible in the original pressure record. MS is influenced by breathing movements (large oscillations) and by other non-determined variables.

In Figure 1B, the Fourier spectrum of the blood pressure shows four components being the most representative of the HR complex with a value of 2.525 Hz. Another complex, with a maximum value of 5.956 Hz corresponds to the vascular activity. In the high frequency zone, there is a small complex of 7.45–7.75 Hz band width (BW), corresponding to secondary vibrations due to valves closure. The zone of very low frequency shows a small complex of 0.0292 – 0.3961 Hz BW, with a maximum value of 0.0417 Hz, corresponding to the vasomotor activity (14).

The contour of module of the wavelet coefficients (Fig. 1C) is shown as two-dimensional time-frequency image, i.e., a time–frequency representation. In the zone of 2 to 4 octaves, i.e., between 4 and 16 Hz, a blue band, corresponding to the secondary vibrations associated to the insula and a clear blue zone, corresponding to the aortic valve closure, is observed. The heart rate frequency (HRF) appears as a red band between 1 and 2 octaves or its equivalent 2 to 4 Hz. Little periodic color changes, within the above-mentioned red band indicate the existence of an AM phenomenon. Additionally, in this figure, the contours of three respiratory episodes between 0 and –2 octaves, or 1 and 0.25 Hz, are observed. Finally, a contour between –3 and –5 octaves, i.e., between 0.125 and 0.031 Hz, corresponding to a minimal vasomotor activity is observed in the very low frequency zone (VLF).

**Blood pressure signal analysis with hemorrhage between 0 and 6% EBV**

In Figure 2, the arrow (H) indicates the initiation of the first hemorrhagic period (time = 10 s), with little and progressive decline of the blood pressure, as the hemorrhage progresses with a decrease in the differential pressure. The MS shows a significant slope, as an indication of a continuous decrease of the differential pressure just after initiating the hemorrhage. This phenomenon is much clearer than in the original plot. Moreover, MS shows very clear breathing activity between other oscillations (Fig. 2A).

The Fourier spectrum (Fig. 2B) shows a relationship with the HR in the 2.05–2.76 Hz BW with a maximum of 2.6375 Hz. The arterial wave frequencies show a 5.10–5.45 Hz BW.
with a maximum of 5.2645 Hz. In the high frequency zone, the displacement to the right side complex is observed with an increase of the BW of 7.65–8.20 Hz with a maximum of 7.8916 Hz. In the VLF zone, we can see another complex with a 0.0104–0.3336 Hz BW, corresponding to the vasomotor activity.

In the contour of module (Fig. 2C), a dramatic and progressive color change (from purple to yellow) of the HR band is observed, indicating a reduction of energy of each pressure pulsation due to the differential pressure decrease, which is correlated with a progressive declination of the MS slope (Fig. 2A). In the low frequency zone, five contours subsequent to breathing activity are observed. In the VLF zone, an increase in the complex borders is observed indicating increasing vasomotor activity.

**Blood pressure signal analysis in the second phase of a graded hemorrhage from 8%–16% EBV**

A decline and a very slow fluctuation of the mean arterial pressure is observed in Figure 3A. Beside the episodes that are related to respiratory movements, the MS shows a progressive decline, as well as the influence of respiration, among other factors. The Fourier spectrum (Fig. 3B), in this hemorrhage period, shows changes in the BW of all the components,
mainly in the very low frequency zone (0.01–0.8 Hz, BW with a maximum in 0.0209 Hz). The HR band shows three peaks: 2.5749, 2.6271, and 2.6791, between 2.45–2.84 Hz. The same occurs in the complex corresponding to the vascular activity with a 5.05 to 5.45 Hz BW. In the HF zone, there is a small complex with a 7.6–8.4 Hz BW.

In the contour of module (Fig. 3C), a change from red to yellow can be observed in the HR band, although less noticeable than the previous phase, indicating a decrease of the signal energy. An increase in the number and complexity of the contours corresponding to breathing cycles is also observed. Finally, an increase in the complexity of the borders is also observed in the VLF zone indicating an increase in the vasomotor activity.

**Fig. 3.** Time–frequency and Fourier analysis of arterial pressure pulsations during the second hemorrhage periods from 8% to 16% of EBV. A, we can observe, beside the periodic cycles of one to four breathing movements, a very slow fluctuation of the mean arterial pressure. The MS shows a progressive decline, as well as the influence of breathing. B shows the Fourier spectrum of hemorrhage with changes in the BW of all the components, especially in the VLF zone. BW of 0.01–0.8 Hz with a maximum in 0.0209 Hz. In the HR band, we can observe three peaks at 2.5749, 2.6271 and 2.6791 between 2.45–2.84 Hz. The same occurs for the complex corresponding to the vascular activity with a 5.05 to 5.45 Hz BW. In the HF zone, a small complex with 7.6–8.4 Hz BW is observed. In the contour of module (C), we can observe a change from red to yellow in the HR band. An increase in the number and complexity of the breathing cycles are observed. Finally in the VLF zone, there is also an increasing complexity of the borders indicating an increase in the vasomotor activity.

**Fig. 4.** Time–frequency and Fourier analysis of arterial pressure pulsations when the magnitude of the hemorrhage is equivalent to 34% of EBV, a recording under steady-state conditions. In A, the arterial pressure oscillations are irregular, and the vasomotor activity (Mayer-waves is pronounced. The MS does not decline as mentioned above, since the bleeding period was over. Nevertheless, this MS shows great instability due to, in part, respiratory influences, and to changes presented in the shape of some pulsations associated to acute pressure decreases. The Fourier spectrum (B) shows increasing activity of VLF with a maximum in 0.0417 Hz, with amplitude value surpassing the HR complex concentrated in 2.2517 Hz. The BW reduction is noticeable in all the components, excepting the VLF complex. The contour module (C) shows, in the HR band, a form red to yellow, indicated by black arrows, representing an abrupt decrease of the energy contained in the signal, associated to changes in the arterial pressure and the shape of some pulsations. Breathing movements are not clearly distinguishable. In the VLF domain [-4, -6] octaves, the intense yellow color corresponds to a pronounced vasomotor activity.

**Blood pressure signal analysis when hemorrhage is equivalent to 34% EBV**

In Figure 4A, the respiratory influences are not clearly distinguishable in the present case, however, low waves of approximately 30 s per cycle, are observed, corresponding to the waves described by Mayer (16) as a result of vasomotor activity (14, 15). We can also observe a decrease in the mean blood pressure. The MS does not follow the tendency of the blood pressure, but presents a slight positive slope, indicating an increase in the energy of the signal.

The Fourier spectrum (Fig. 4B) shows a notorious decrease in the BW at all complexes excluding that of the VLF. In the
VLF zone, it shows an activity with a maximum of 0.0417 Hz with larger amplitude than the HR complex and centered in 2.2517 Hz. In 4C, the HR band between 1 to 1.5 octaves is shown with pronounced irregularities, which are due to an unknown factor, and are indicated by black arrows. Additionally, a slight increase of the band color towards red is observed, a phenomenon that agrees with the positive slope of MS. Here, the closure of the aortic valves has now a yellow to red coloration, indicating an increase in the energy of the valvular activity. In the breathing complexes zone it is possible to identify the breathing cycles, observing a series of irregular contours. In addition, it is possible to observe, in the zone of 0 to –1 octaves, a small green-yellow spot associated to two irregularities of the pressure signal occurred between 25 and 30 s after registration.

An extended yellow spot, between –4 to –5 octaves (i.e., between 0.06 to 0.03 Hz) appears, indicating an important increase of the vasomotor activity, a phenomenon which agrees with the peak observed in the VLF zone in the Fourier spectrum (Fig. 4B).

**Blood pressure signal analysis in steady-state conditions, with 66% EBV**

In Figure 5, the main features of this hemorrhagic phase, is the presence of shock pressure levels whose range is within 40–60 mmHg, with a marked decrease in HR. In addition, alterations in the shape of some pulses with rapid decrease of blood pressure are observed. As in the previous phase, it is not possible to observe breathing movements. In the MS, a slight decline and great oscillations can be observed. The Fourier spectrum shows a decrease in frequency values in all components, and a reduction of the amplitude of the very low frequency complexes.

In the contour of module (Fig. 5C), we can observe a decreasing frequency in octaves of all bands, agreeing with the left displacement of the complexes in Fourier spectrum. In the band of the valvular activity, the red-yellow color is changed to green-yellow, corresponding to decreasing energy of the signal with regards to the previous case (Fig. 4C). In this same band, a yellow spot appears between 5 and 6 s, coincident with a severe transitory decrease of blood pressure. This phenomenon produces an interruption of the red HR band and a small green-yellow spot in the low frequency zone around –2 octaves. In the HR band, a series of discontinuities in the red color are observed due to the strong decrease of the original record. In the low frequency zone, as in the previous case, (Fig. 4C) no contours associated to the breathing cycles are observed. The decrease of the yellow spot corresponding to the vasomotor activity observed in Fig. 4C is notorious, which is confirmed by the decrease of the complex amplitude corresponding to the Fourier spectrum.

**DISCUSSION**

In previous studies (6–8) we found that both the discrete wavelet transform (DWT), such as the CWT, are excellent tools for the qualitative analysis of the time–frequency behavior of the cardiovascular system. The CWT analysis of an AM phenomenon present in the arterial pressure, previously described using DWT (6), allowed the identification of MS. This MS curve provides information on the energy contained in the signal in every instant, resulting in a useful tool to analyze the changes that occur during hemorrhage. The MS is able to detect changes in the signal immediately after the hemorrhage starts, even when no changes are observed in the original signal (Figures 2A and 3A). Additionally, the breathing movements and eventually other unknown factors influence the MS curve. The CWT, under the same energy reduction phenomenon, shows a drastic color change in the HR band (Figures 2C and 3C). The CWT also gives information about breathing movements and the valvular activity of the left
ventricle. The Fourier spectrum does not detect these transient activities. The CWT describes a VLF vasomotor activity, when this activity is very intensive, the Fourier spectrum shows the same result.

The MS and the CWT show a clear recovering of the energy contained in the signal during the post-hemorrhage rest period, indicated by a positive decline of the MS (Fig. 4A), and by an increase of the red color intensity in the HR band (Fig. 4C), although neither the original signal nor Fourier spectrum perceive this phenomenon. The CWT also shows, under the same conditions, an increasing valvular activity, expressed in a color change in the respective band. The CWT also results to be efficient to evidence vasomotor activity of VLF, which seems to be more pronounced as the hemorrhage advances, a phenomenon which is confirmed by the Fourier spectrum, when such vasomotor activity is intensive.

When the amount of hemorrhages is equivalent to 66% EBV, systemic arterial pressure reaches shock levels, with a HR decrease close to sinus rhythm values and a transitory modification of the pulse wave shape, mainly in the posterior segment to the valve closure is observed, associated to a temporal decrease of the diastolic pressure, followed by a ventricular contraction, unable to produce an adequate increase of the systolic pressure (Fig. 5). It is also notorious that at this stage (~66% EBV), almost all of the vasomotor activity decrease is associated with the disappearance of Mayer’s waves, the decrease of the VLF complex in the Fourier spectrum, and the attenuation of a yellow spot in the very low frequency zone of the contour of module. This phenomenon could be explained by the exhaustion of the sympathetic response, as related with the irreversibility of the shock syndrome (17). In all experiments, we did not observe the expected reflex increase in heart rate during hemorrhage, phenomenon probably due to the effects of anesthesia, different results should be expected when experimental animals are submitted to progressive hemorrhages in conscious animals (18, 19). Although hyperventilation of animals before hemorrhage improves cell and organ function (20), we didn’t find any influences on eventual cardiovascular effects of heparin. Both the MS and the CWT detect small changes in the form of the pulse waves, expressed in MS oscillations, and by color changes in the bands of module. For example, in Figure 6, we show an amplification of the XX’ interval of Figure 5, and its corresponding graph of the wavelet coefficients module. In Figure 6A, between 5 and 6 s, we can observe a sudden decrease of the arterial pressure followed by a small ventricular contraction that is unable to reach the mean level of the systolic pressure. This event is reflected in the module by the notorious color changes. A second accident, apparently similar to the previous one, occurred between 14 and 15 s, showing a very different time-frequency behavior, since almost imperceptible changes in the wave shape, are visualized as intense color changes in the module. For example, a small change in the pulse form at 11 s is reflected by a color change in the aortic valve zone (white arrow). These isolated pressure accidents may be interpreted as premature interpolated ventricular contractions without a compensatory pause (21).

In summary, these results allow us to conclude that the CWT give more information than the Fourier spectrum in the study of transient phenomena occuring during acute hemorrhage. Effectively, while the Fourier transform is a "static" frequency analysis, the wavelet transform yields a "dynamical" time-frequency analysis of arterial pulsations, whose spectrum includes the aortic valves oscillations as well as the slow vasomotor activity of vascular smooth muscle contraction, and in consequence, is an holistic approach of cardiovascular rhythmicity, both in normal and pathological conditions. In addition, CWT allows to obtain a MS curve giving information related to these studies. Some of the phenomena found in the present work, could be useful to understand the shock pathophysiology in future studies, and probably may help to support the precious diagnosis of an internal hemorrhage. Therefore, this analysis could be useful as an endpoint to guide resuscitation efforts.

ACKNOWLEDGMENTS

Our gratitude is expressed to Dr. Luis Caú, of the Statistics Department, Faculty of Physics and Mathematics, University of Concepcion, for his critical comments and discussion. We also thank Sylvia Gutiérrez and Claudia Contreras for their excellent technical assistance. We are also grateful to the Editor of this journal and unknown referees for their invaluable suggestions.
SHOCK FEBRUARY 2001

This research was supported by Grant 95-1569-1 from DIRECCIÓN DE INVESTIGACION, University of Concepción, and by grant 197/008 from FONDECYT.

REFERENCES


APPENDIX

An outline of CWT

The aim of CWT is to provide a two-dimensional visual interpretation of the original signal. The CWT is a scale-time decomposition, which is well adapted to the model and to study the time-frequency behavior, which enables us to analyze transients, or else, to detect the effects of impulses and discontinuities of the original signal (H1).

The wavelet transform has already been applied successfully in many fields: acoustics, sound processing, image processing, seismology, mechanics, and the study of fractals. However, only few applications to physiological signals have been reported (4-6, 12, 13).

Definitions

The CWT of a real signal f(t) with regards to an analyzing wavelet ψ(t) may be defined as:

$$\text{CWT}(s, τ) = \frac{1}{\sqrt{s}} \int_{\mathbb{R}} f(t) \psi^*(\frac{t - τ}{s}) dt, \quad s > 0, \tau \in \mathbb{R} \quad (1)$$

where $\psi^*$ denotes the complex conjugate of $\psi$, defined on the open "time and scale" as half-plane H. It is convenient to utilize a somewhat unusual coordinate system on H, with $\tau$-axis ("dimensionless time") running to the right and the $s$-axis ("scale") running downward. The $\tau$-axis faces downward because the small scales correspond, roughly speaking, to high frequencies above low frequencies.

Equation (1) can also be written in terms of Fourier transform $\hat{f}(ω)$, $\psi(ω)$ do f(t) and $\phi(t)$ respectively, as:

$$\text{CWT}(s, τ) = s^{-1/2} \int_{\mathbb{R}} \hat{f}(ω) \hat{ψ}(ω) e^{jωτ} dω \quad (2)$$

Where we impose on $ψ$ the admissibility condition:

$$K_ψ = 2π \int_{\mathbb{R}} \left| \hat{ψ}(ω) \right| dω < \infty \quad (3)$$

If $ψ^*(ω)$ is differentiable, as it is assumed here, this implies that $ψ^*(0) = 0$, i.e. $ψ$ is of mean zero.

The main motivation for the admissibility condition is that it implies the weak convergence of:

$$\lim_{s \to 0^+} \int_{\mathbb{R}} \left| f, ϕ_{s, s} > ϕ_{s, s} \right| \frac{ds}{s^2} \quad (4)$$

where $ϕ_{s, s} = \frac{s}{\sqrt{2π}} ϕ(\frac{t - τ}{s})$ and $<, >$ is the inner product in Lebesgue space $L^2(\mathbb{R})$, space of the finite energy signals. This latter expression is related with the reconstruction formula:

$$\hat{f}(ω) = K_ψ \int_{\mathbb{R}} \hat{ψ}(ω) \hat{ψ}(ω) e^{jωτ} \left( \frac{1}{\sqrt{2π}} ϕ(\frac{t - τ}{s}) \right) dτ \quad (5)$$

which is exactly Calderón's reproducing identity.

The wavelet coefficients of signal $f(t)$ are the inner products in $L^2$ space defined by:

$$\left< f, ϕ_{s, τ}, > \right| = \int_{\mathbb{R}} f(t) ϕ_{s, τ}(t) dt, \quad s > 0, τ \in \mathbb{R} \quad (6)$$
If in a certain zone of the signal high frequencies are present, the coefficient wavelet are big; if the signal is almost constant, then the coefficient values are small (because the wavelets are of null media). Likewise, the magnitude of the wavelet coefficient indicates the spectral content of the signal for a certain s-scale. When the s parameter varies, the wavelets cover different frequency ranges; at high values of s-scale, low frequency correspond (or at a greater scale of ϕ). By changing the parameter τ, we can move the localization center in the time axis being able to cover the entire signal (each wavelet is localized around \(t = \tau\)).

The relevance of a continuous wavelet analysis will depend greatly on the properties of the analyzing wavelet \(\psi\). Two usual choices are the Morlet's wavelet (a modulated Gaussian) or the Mexican hat (the second derivative of a Gaussian). In this paper the Morlet wavelet is used:

\[
\psi(t) = \pi^{-1/4} \omega_0^{1/2} e^{-\omega_0^2 t^2 / 2}, \quad (\omega_0 > 5),
\]

then, \(\tilde{\psi}(\omega) = \pi^{-1/2} e^{-\omega_0^2 \omega^2 / 2}\).

**Implementation of CWT using FFT**

Equation (1) could be expressed as a convolution to use FFT as an optimum calculation algorithm to carry out this operation. Convolutions in time constitute a simple multiplication in frequency from the inverse Fourier transform (IFT), from which we directly obtain the required wavelet coefficients. Effectively, using Equation (1) and the Morlet complex wavelet:

\[
\psi(t) = e^{i\omega_0 t} e^{-\omega_0^2 t^2 / 2}
\]

we obtain, by applying translation and dilations to the wavelet:

\[
\psi(t, \tau, s) = e^{i\omega_0 (\tau - t)} e^{-\omega_0^2 (t - \tau)^2 / 2s^2}
\]

Reordering this equation in such a way that the resulting function as a product of convolution in time:

\[
\text{CWT}(t, \tau) = s^{-1/2} \int_{\mathbb{R}} \tilde{f}(\omega) e^{-i\omega_0 (\tau - t)} e^{-\omega_0^2 \omega^2 / 2s^2} \, d\omega
\]

that is:

\[
\text{CWT}(t, \tau) = s^{-1/2} \text{conv} [\tilde{f}(\omega), \psi(\omega, \tau, s)]
\]

Then, by discretizing and applying the direct Fourier transform to each function, and then applying the inverse operation, we finally obtain:

\[
\text{CWT}(s) = \text{IFFT} \left( s^{1/2} \cdot \text{FFT} [\tilde{f}(\omega)] \cdot \text{FFT} [\psi(\omega, \tau, s)] \right)
\]

Note that \(\tau\) parameter is implicit in the convolution, therefore, the CWT depends only on the s-scale parameter. This parameter constitutes the continuous transform parameter. The usual selection of s-scale is realized in potencies of 2, to which a range of frequencies divided in octaves is associated. In addition each octave is divided in voices, to give continuity to the graphic representation. A greater number of voices per octaves imply a better visualization of the transformation, as well as, the graduation in octaves allows the pass of graduation in frequency, so that for each division the frequencies are doubled.